

## M1

1. A triangle  $ABC$  with  $AC = 20$  is inscribed in a circle  $\omega$ . A tangent  $t$  to  $\omega$  is drawn through  $B$ . The distance of  $t$  from  $A$  is 25 and that from  $C$  is 16. If  $S$  denotes the area of the triangle  $ABC$ , find the largest integer not exceeding  $S/20$ .
2. In a parallelogram  $ABCD$ , a point  $P$  on the segment  $AB$  is taken such that  $\frac{AP}{AB} = \frac{61}{2022}$  and a point  $Q$  on the segment  $AD$  is taken such that  $\frac{AQ}{AD} = \frac{61}{2065}$ . If  $PQ$  intersects  $AC$  at  $T$ , find  $\frac{AC}{AT}$  to the nearest integer.
3. In a trapezoid  $ABCD$ , the internal bisector of angle  $A$  intersects the base  $BC$  (or its extension) at the point  $E$ . Inscribed in the triangle  $ABE$  is a circle touching the side  $AB$  at  $M$  and side  $BE$  at the point  $P$ . Find the angle  $DAE$  in degrees, if  $AB : MP = 2$ .
4. Starting with a positive integer  $M$  written on the board, Alice plays the following game: in each move, if  $x$  is the number on the board, she replaces it with  $3x + 2$ . Similarly, starting with a positive integer  $N$  written on the board, Bob plays the following game: in each move, if  $x$  is the number on the board, he replaces it with  $2x + 27$ . Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of  $M + N$ .
5. Let  $m$  be the smallest positive integer such that  $m^2 + (m + 1)^2 + \dots + (m + 10)^2$  is the square of a positive integer  $n$ . Find  $m + n$ .
6. Let  $a, b$  be positive integers satisfying  $a^3 - b^3 - ab = 25$ . Find the largest possible value of  $a^2 + b^3$ .
7. Find the number of ordered pairs  $(a, b)$  such that  $a, b \in \{10, 11, \dots, 29, 30\}$  and  $\text{GCD}(a, b) + \text{LCM}(a, b) = a + b$ .
8. Suppose the prime numbers  $p$  and  $q$  satisfy  $q^2 + 3p = 197p^2 + q$ . Write  $\frac{q}{p}$  as  $l + \frac{m}{n}$ , where  $l, m, n$  are positive integers,  $m < n$  and  $\text{GCD}(m, n) = 1$ . Find the maximum value of  $l + m + n$ .
9. Two sides of an integer sided triangle have lengths 18 and  $x$  where  $x < 100$ . If there are exactly 35 possible integer values  $y$  such that  $18, x, y$  are the sides of a non-degenerate triangle, find the number of possible integer values  $x$  can have.
10. Consider the 10-digit number  $M = 9876543210$ . We obtain a new 10-digit number from  $M$  according to the following rule: we can choose one or more disjoint pairs of adjacent digits in  $M$  and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from  $M = 9876543210$ , by interchanging the 2 underlined pairs, and keeping the others in their places, we get  $M_1 = 9786453210$ . Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from  $M$ .
11. Let  $AB$  be a diameter of a circle  $\omega$  and let  $C$  be a point on  $\omega$ , different from  $A$  and  $B$ . The perpendicular from  $C$  intersects  $AB$  at  $D$  and  $\omega$  at  $E (\neq C)$ . The circle with centre at  $C$  and radius  $CD$  intersects  $\omega$  at  $P$  and  $Q$ . If the perimeter of the triangle  $PEQ$  is 24, find the length of the side  $PQ$ .

12. Given  $\triangle ABC$  with  $\angle B = 60^\circ$  and  $\angle C = 30^\circ$ , let  $P, Q, R$  be points on sides  $BA, AC, CB$  respectively such that  $BPQR$  is an isosceles trapezium with  $PQ \parallel BR$  and  $BP = QR$ . Find the minimum possible value of  $\frac{2[ABC]}{[BPQR]}$  where  $[S]$  denotes the area of any polygon  $S$ .
13. Let  $ABC$  be a triangle and let  $D$  be a point on the segment  $BC$  such that  $AD = BC$ . Suppose  $\angle CAD = x^\circ$ ,  $\angle ABC = y^\circ$  and  $\angle ACB = z^\circ$  and  $x, y, z$  are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of  $\angle ABC$  in degrees.
14. Let  $x, y, z$  be complex numbers such that

$$\begin{aligned}\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} &= 9 \\ \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &= 64 \\ \frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} &= 488\end{aligned}$$

If  $\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} = \frac{m}{n}$  where  $m, n$  are positive integers with  $\text{GCD}(m, n) = 1$ , find  $m + n$ .

15. Let  $x, y$  be real numbers such that  $xy = 1$ . Let  $T$  and  $t$  be the largest and the smallest values of the expression

$$\frac{(x+y)^2 - (x-y) - 2}{(x+y)^2 + (x-y) - 2}.$$

If  $T+t$  can be expressed in the form  $\frac{m}{n}$  where  $m, n$  are nonzero integers with  $\text{GCD}(m, n) = 1$ , find the value of  $m + n$ .

16. Let  $a, b, c$  be reals satisfying

$$3ab + 2 = 6b, \quad 3bc + 2 = 5c, \quad 3ca + 2 = 4a.$$

Let  $\mathbb{Q}$  denote the set of all rational numbers. Given that the product  $abc$  can take two values  $\frac{r}{s} \in \mathbb{Q}$  and  $\frac{t}{u} \in \mathbb{Q}$ , in lowest form, find  $r + s + t + u$ .

17. For a positive integer  $n > 1$ , let  $g(n)$  denote the largest positive proper divisor of  $n$  and  $f(n) = n - g(n)$ . For example,  $g(10) = 5, f(10) = 5$  and  $g(13) = 1, f(13) = 12$ . Let  $N$  be the smallest positive integer such that  $f(f(f(N))) = 97$ . Find the largest integer not exceeding  $\sqrt{N}$ .

18. Let  $m, n$  be natural numbers such that

$$m + 3n - 5 = 2\text{LCM}(m, n) - 11\text{GCD}(m, n).$$

Find the maximum possible value of  $m + n$ .

19. Consider a string of  $n$  1's. We wish to place some  $+$  signs in between so that the sum is 1000. For instance, if  $n = 190$ , one may put  $+$  signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If  $a$  is the number of positive integers  $n$  for which it is possible to place  $+$  signs so as to get the sum 1000, then find the sum of the digits of  $a$ .

20. For an integer  $n \geq 3$  and a permutation  $\sigma = (p_1, p_2, \dots, p_n)$  of  $\{1, 2, \dots, n\}$ , we say  $p_l$  is a *landmark point* if  $2 \leq l \leq n-1$  and  $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$ . For example, for  $n = 7$ , the permutation  $(2, 7, 6, 4, 5, 1, 3)$  has four landmark points:  $p_2 = 7, p_4 = 4, p_5 = 5$  and  $p_6 = 1$ . For a given  $n \geq 3$ , let  $L(n)$  denote the number of permutations of  $\{1, 2, \dots, n\}$  with exactly one landmark point. Find the maximum  $n \geq 3$  for which  $L(n)$  is a perfect square.
21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If  $N$  is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of  $N$ .
22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence  $0, 1, 1, 0, 0, 1, 1, 1$  is friendly. Let  $F_n$  denote the number of friendly binary sequences with  $n$  terms. Find the smallest positive integer  $n \geq 2$  such that  $F_n > 100$ .
23. In a triangle  $ABC$ , the median  $AD$  divides  $\angle BAC$  in the ratio  $1 : 2$ . Extend  $AD$  to  $E$  such that  $EB$  is perpendicular  $AB$ . Given that  $BE = 3$ ,  $BA = 4$ , find the integer nearest to  $BC^2$ .
24. Let  $N$  be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If  $N = 100a + b$ , where  $a, b$  are positive integers less than 100, find  $a + b$ .

## Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12
Answer	10	67	60	10	95	43	35	32	82	88	08	03

QNo	13	14	15	16	17	18	19	20	21	22	23	24
Answer	59	16	25	18	19	70	09	03	74	11	29	81

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